Allowable Working Stresses Under Impact

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This paper discusses the nature of failure under impact, and indicates how the factor of safety under these conditions is determined by the properties of the material employed.

The proper factor of safety to employ in a given case is mainly dictated by two conditions: First, by the degree of reliability with which the stresses are calculated (data concerning the possible external loading, reliability of the fundamental theoretical assumptions, and correct representation of actual working conditions), and second, by the degree of certainty respecting the similarity of the mechanical properties of the material with those of the specimens used in the laboratory tests, (homogeneity of the material, sufficient similarity of working conditions of the material in service and under test). With respect to impact, however, both of these conditions are very complicated, as the calculation of the theoretical stresses can be made only in the simplest cases, and the conditions under which tests are made in the laboratory are usually those obtaining at static speeds, and very far from the conditions existing in actual service. Therefore, in order to determine the proper safe working stresses, it is necessary to study both sides of the question in detail.

1—Methods of Calculating Stresses Under Impact

At the present time there are two ways of analyzing impact stresses: the theoretical, or, properly speaking, the dynamic, and the empirical, or static, with a correction made for dynamical conditions.

The first method gives an accurate solution of the problem only in exceptionally simple cases, such as elementary problems of longitudinal impact of bars with rounded ends, impact of elastic spheres, impact of a sphere falling on an elastic beam, etc. This method is of but little practical importance. In the majority of cases other than those mentioned it is necessary to resort to approximate solutions, based upon the assumption of similarity of static and impact stresses and neglecting the time for propagation of the elastic wave (kinetic-energy method). In longitudinal impact, this assumption is equivalent to the admission that at any given instant the whole bar is affected by a homogeneous state of stress. It is difficult to ascertain a priori the order of error, particularly because of the fact that the impact stresses are to a considerable degree affected by the local conditions of the impinging surfaces. These conditions for a given impact loading may vary, depending on details of the particular application (e.g., the state of the edges of a rail joint during impact of a rolling wheel), and resulting in a different magnitude of stress. Therefore the reliability of the theoretical calculation of the stresses is very low.

The second method is applied in cases where series of impacts cause oscillation of the loading about a certain mean value corresponding to the static conditions. For instance, the loading exerted by wheels of a moving train on the rails or on a bridge truss, by the wheels of a truck on the highway pavement, etc. are of this kind. In such cases it is customary to judge the impact character of the loading by a dynamic coefficient greater than unity, expressing the ratio of the maximum impact load to the equivalent load under static conditions. The value of this coefficient can be determined in some cases in a purely empirical way, and our knowledge in this field is constantly increasing. However, in many cases we still have to be satisfied with arbitrary assumptions about the dynamic coefficients by employing merely general considerations.

In view of the uncertainty of calculation of the impact stresses and of the variety of solutions obtained, it is not possible to make any general statement concerning the allowable working stresses; therefore, in the following there will be considered only the case in which the impact stresses, calculated in one way or another, are checked experimentally and are consequently reliable.

As far as the actual methods of checking are concerned, difficulties arise because of the short duration of the impact (usually of the order of a few thousandths of a second) and of the wave character of the stress distribution. This checking will be treated in detail later.

2—Determination of the Mechanical Properties in the Laboratory

In considering the mechanical properties of materials, the fundamental problem may be stated as follows: Is it permissible in calculating the allowable working stresses to use as a basis the yield point and the tensile strength of the material obtained under static conditions of loading?

The question has to be stated in this way for the reason that purely impact testing for rupture as usually carried out does not afford any means of determining directly the behavior of the metal during the impact. It is necessary to integrate the forces throughout the whole region of deformation, or to calculate the amount of the energy of deformation, which, however, cannot be used directly as characteristics for structures. Consequently there arises the following question: What relation exists between the dynamic and the static properties (mainly, the yield point and tensile strength), which of them is the larger, and how much? The answer can be given only with the help of scientific investigations which are of a complicated nature but are nevertheless available in large numbers.

In studying the influence of the variable speeds of static loading, we are enabled to predict the change of properties at impact speeds. Long ago it was noticed that in the plastic deformation of solid bodies internal friction increases with the speed, approaching the friction of viscous liquids; it is therefore natural to expect

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an increase of resistance to plastic deformation with an increase of speed. On this property was based the further one of "relaxation" (at constant deformation) which was studied by Maxwell, and also one generally known as the creep of metal (at constant stress). The conception of Schmid and Polanyi gives perhaps the clearest physical idea regarding these properties. They propose to differentiate between "thermal" and "athermal" plasticity. It is assumed that in the process of plastic deformation the heat motion of the atoms (thermal plasticity) continuously reduces the strengthening caused by cold working (athermal plasticity); in the case of short-duration loading (high speed) or loading at low temperatures, the reduction of the strengthening is slackened, and it is necessary to supply more energy for the same deformation.\footnote{A. Dinnik, Izv. Kieff, Poly. Inst. (Trans. Poly. Inst. of Kieff), vol. 10 (1913), pp. 421-462, 547-580.}

The nearer the melting temperature of the metal is to the room temperature, the more evident is this influence of the speed because of the larger mobility of the atoms. Therefore, for example, lead is more susceptible to speed than iron.\footnote{Proc. Inst. M. E., 1910, p. 715.} Now, considering the speeds of impact, we have to expect this phenomenon to manifest itself to a still greater extent. Here, however, appear the experimental difficulties mentioned above.

One of the simplest, but at the same time one of the crudest methods of measuring the impact force during rupture of a specimen in the laboratory is to calculate the average strength of the specimen; for this purpose it is sufficient to divide the measured energy of deformation of the specimen by the total elongation of the latter. Comparing the stress-strain diagram obtained under dynamic loading with one obtained under static conditions, it is possible to obtain a preliminary idea regarding the influence of the speed. According to the experiments by Blount, Kirkaldy, and Sankey,\footnote{E. Siebel and A. Pomp, Mitt. K.-W. Inst. für Eisenforschung, vol. 10 (1928), p. 63.} this influence for various steels at high impact speeds (height of fall of the blow-imparting mass, or tup, 12 meters) was expressed by ratios ranging from 1.24 to 1.55, and from 1.23 to 1.37 according to experiments by Körber and Sack.\footnote{F. Körber and B. Sack, Mitt. K.-W. Inst. für Eisenforschung, vol. 4 (1922), p. 11.}

The next difficult problem is the measuring of the elastic limit and the yield point for impact. This is easier to accomplish because of the fact that up to the appearance of the first permanent deformations the specimen obeys the laws of elasticity, and its deformations can be considered as a measure of the stresses. Of the various methods used for this purpose, only two employed by the author will be considered here.

During one investigation,\footnote{N. Davidenkoff and K. Yurieff, First communication of the NIATM, Zurich, 1930, p. 231.} the following arrangement shown in Fig. 1 was used. The specimen \( A \) was inserted in the two tups \( C \) and \( D \) of the Amser testing machine in such a way that the lower tup \( D \) was suspended from the upper tup \( C \) not through the specimen, as is done in usual testing, but with the wires

\[ A_{D} / p = A_{S} / \gamma = \eta \]

where \( A_{S} \) is the original area of the section in the cylindrical part and \( p \) is the tensile strength of the material. From static tests on similar specimens it was possible to determine the same ratio for low speeds and then to make a comparison.

Experiments performed on six different varieties of annealed carbon steel showed that the ratio \( \gamma \) for impact loading is always larger than for static loading, the average of the two being 1.25, the ratio varying from 1.10 to 1.43.

Finally, A. Dinnik\footnote{A. Dinnik, Izv. Kief, Poly. Inst. (Trans. Poly. Inst. of Kieff), 1909 (in Russian).} investigated the impact of steel spheres against a steel block with a plane polished surface; by observing the instant that the first permanent deformations appeared, namely, aging, causing a new rise in strength, and employing the exact theory of H. Hertz, he calculated the corresponding stresses. By comparing these with the corresponding stresses in static loading, he found that the dynamic yield point exceeds the static in a ratio of at least 1.55.

The foregoing values, together with data from other series of investigations, are given in Table 1.
All the experimental data show a considerable rise of the yield point at impact with the exception of those recently reported by Guest, which are of opposite character. Guest measured and recorded the elastic deformation during longitudinal impact of two long iron bars (11 ft and 23 ft), and using this deformation, he checked the correctness of the methods employed in calculating the stresses. As the yielding limit, he assumed the stress at which the coefficient of recovery after the impact (relative height of rebound of the striking bar) started sharply to fall. This information appears does not depend on the speed; but if the stress (31 kg per sq mm) but slightly exceeded (less than 1 per cent) the static value of the elastic limit, the coefficient of recovery after the impact (relative height of rebound) started to fall sharply. This stress (31 kg per sq mm) but slightly exceeded (less than 1 per cent) the static value of the elastic limit. It seems to be possible to reconcile these contradictions. In all experiments mentioned above, rather large deformations were used as criteria of passing beyond the yielding limit, such as the Lüders lines in the author's experiments or an apparent deformation changing the refraction of light (Dinnik); but Guest in his experiments with the most accurate measurements (=0.0001 in.) discovered no change in the dimensions of the bar after passing beyond the yield point. Therefore it is possible to believe that the stress at which the first exceedingly small permanent deformation appears does not depend on the speed; but if the determination of the elastic limit or the yield point is connected with a definite value of the permanent deformation or with a definite degree of its external expression (Lüders lines), the difference from the data given in Table 1 is then apparent, because the short duration on the blow does not permit the deformation to sufficiently develop.

For the designer, the stresses beyond the yielding limit are dangerous only in connection with the above-mentioned external effects, and therefore the numerical results of Table 1 remain valid.

Finally, in measuring the strength of a material under impact it is necessary to use extremely elaborate experimental methods, which will permit determining fully the whole stress-strain diagram in impact. Many procedures have been proposed for this purpose, but none of them received has been employed in more than a single investigation. They all fall into one of two classes:

1. The curve of the motion of the tups that ruptures the specimen is recorded as a function of time, \( s = f(t) \), and by differentiating this curve twice, the acceleration, \( ds^2/dt^2 \), and consequently the stress of the specimen, is obtained. (See Plank,\(^{17}\) Ermendorf,\(^{18}\) Gett,\(^{19}\) Homger,\(^{20}\) Seehase,\(^{21}\) Schwinning and Matthias,\(^{22}\) Köber and Storp,\(^{23}\) and Yamada.\(^{24}\) This method has the disadvantage that considerable errors are unavoidable in the process of double graphical differentiation.

2. Elastic self-recording dynamometers are used: for example, in measuring the elongation of a steel bar (Moore,\(^{25}\) Meyer\(^{26}\) ); mutual compression of spherical lenses (Kirner,\(^{27}\) Davidenko\(^{28}\) ); deformation of piezoquartz (Prince Galitzyn,\(^{29}\) Kluge and Lineck\(^{30}\) ). The difficulty in using dynamometer methods lies in the necessity of providing the elastic system of the dynamometer with a natural-vibration frequency several times smaller than the duration of that part of the investigation during which the change of the force must be correctly recorded. Apparently the use of piezoquartz gives the best results, although for recording the elastic charge, which varies quickly, it is necessary to have an oscillograph of almost negligible inertia. This condition can be met only by using a cathode oscillograph.

Adopting this latter method, an experiment in measuring the impact resistance of specimens was carried out in the author's laboratory.\(^{31}\) Fig. 3 is a schematic diagram showing the use of the oscillograph. The cathode beam was simultaneously affected by two fields giving transitory motion in two planes perpendicular to each other, namely, an electric field, obtaining its potential from the quartz crystals at \( D \), and a magnetic field from special spools with large self-induction introduced automatically.

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TABLE 1 RATIOS OF RISE OF YIELDING LIMIT AT IMPACT

<table>
<thead>
<tr>
<th>Authority</th>
<th>Year</th>
<th>Material</th>
<th>Ratio of rise</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>B. Hopkinson(^{17})</td>
<td>1905</td>
<td>Steel</td>
<td>1.68</td>
<td></td>
</tr>
<tr>
<td>N. Nemiloff(^{18})</td>
<td>1910</td>
<td>Steel</td>
<td>1.90</td>
<td></td>
</tr>
<tr>
<td>N. Davidenko(^{19})</td>
<td>1913</td>
<td>Steel</td>
<td>1.23-1.44</td>
<td></td>
</tr>
<tr>
<td>E. Meyer(^{20})</td>
<td>1927</td>
<td>Iron</td>
<td>1.90-1.60</td>
<td></td>
</tr>
<tr>
<td>K. Junie(^{21})</td>
<td>1927</td>
<td>Steel</td>
<td>1.10-1.43</td>
<td>Relative (not absolute) rise of yield point</td>
</tr>
</tbody>
</table>

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\(^{18}\) N. Nemiloff, The Engineer (Kieff), 1910, nos. 11 and 12 (in Russian).
\(^{19}\) E. Meyer, Forschungsarbeiten, V.D.I., no. 295, 1927.
\(^{23}\) P. Breuil, Revue de Mecanique, 1909.
\(^{24}\) H. Seehase, Forschungsarbeiten, V.D.I., no. 182, 1915.
\(^{25}\) W. Schwinning and K. Matthias, Deutscher Verbund für Materialprüfung der Technik, 1927, no. 78.
\(^{27}\) R. Yamada, Science Reports, Tohoku University, vol. 17 (1928), p. 179.

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FIG. 3 DIAGRAM SHOWING USE OF CATHODE OSCILLOGRAPH FOR RECORDING IMPACT FORCES

A = Ammeter
B = Tup
C = Specimen
D = Piezoquartz (4 crystals)
E = Cathode oscillograph (P, photographic film; H, diaphragm)
F = Glowing wire
G = Coil deflecting the cathode beam (---) from aperture in diaphragm H before impact (for protecting R)
K = Coil connected with induction coil I and introduced at the instant of impact, imposing on the cathode beam a transitory motion along the time axis
L = Electrodes receiving potential from charge of the piezoquartz and moving beam along the axis of charges (forces). It is possible to work with one or with two pairs of plates
M = Mechanical arrangement at tup B, grounding the coil G before B begins to fall and adjusting cathode beam in its normal position.

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\(^{29}\) E. Meyer, Forschungsarbeiten, V.D.I., no. 295, 1927.
\(^{30}\) J. Kirner, Ibid., no. 88, 1910.
\(^{32}\) By the physicist W. J. Feoktistoff with assistance of Mr. Klaustrophysics.
ject a diagram on the photographic film, $P$, the ordinates of this diagram being proportional to the pressure on the quartz at each instant, while the abscissas represented the time to some (exponential) scale. Fig. 4 shows the diagrams obtained for brass and lead specimens. The results of these preliminary experiments showed the value of the method, and at present work is under way on the construction of more perfect apparatus for testing specimens in rupture.

In spite of the great variety of methods employed, the number of quantitative results obtained is very small, because most of the experiments were those in which the specimens were bent but not ruptured. All of them testify a general rise of the impact curve as compared with the static curve, this being more pronounced in the first half of the diagram than in the latter. Some of the ratios for metals are given in Table 2.

![Fig. 4 Impact Diagrams for Brass (a) and Lead (b) Specimens](image)

**TABLE 2** **RATIOS OF RISE OF STRENGTH OF MATERIAL UNDER IMPACT**

<table>
<thead>
<tr>
<th>Authority</th>
<th>Year</th>
<th>Material</th>
<th>Yield Point</th>
<th>Strength</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plank</td>
<td>1912</td>
<td>Steel</td>
<td>1.33</td>
<td>1.46</td>
<td></td>
</tr>
<tr>
<td>Meyer</td>
<td>1927</td>
<td>Steel</td>
<td>1.12</td>
<td>1.06</td>
<td>Initial speed of impact</td>
</tr>
<tr>
<td>Seehase</td>
<td>1915</td>
<td>Copper</td>
<td>1.13</td>
<td>1.25</td>
<td>Compression</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Brass</td>
<td>1.01</td>
<td></td>
<td>Stressors referred to equal deformations</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Iron</td>
<td>1.11</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The ratio of rise for strength is not so large as for the yield point, but is large enough to be observed in all experiments (except perhaps those on brass).

The general and indisputable conclusion to be drawn from the foregoing data is as follows: The tensile strength of the material and especially the yield point are higher under impact than under static conditions; however, the amount of rise of the strength and yield point is difficult to state because of lack of agreement of the experimental data available at the present time.

3—SELECTION OF ALLOWABLE WORKING STRESSES

It is a question whether or not it is possible to base the determination of factors of safety upon the increased values of the yield point and strength under impact instead of using the static values. It seems to the author that the answer must be negative for the following reasons:

1. The static loading is imposed by a force (weight, pressure of water, etc.), while the impact is by a definite amount of kinetic energy and fundamentally different. Any local inhomogeneity of the material (blow holes, etc.), and the local weakening of any cross-section, leads to a decrease in strength of the specimen in the same ratio, while the work of deformation of such a bar decreases to a much greater extent because of the decrease in its total elongation.\(^1\)

2. In the case of repeating the same impact beyond the elastic limit (the author does not have in mind here the typical repeated loading encountered in problems of fatigue of metals), the permanent deformation received at the first impact may increase because of the increased duration of the activity of the same force, while under static load the permanent deformation does not depend on the number of applications of the force. This fact diminishes somewhat the favorable effect of the rise of strength occasioned by impact.

3. Incidentally, increases of the loading under impact, as compared with those determined by calculations, are more probable than under conditions of static action of the load, because of the larger number of factors which may influence the magnitude of the impact load.

All this can be considered as an argument for using the same allowable stresses in designing members of structures subjected to impact as are used in static calculations.

In making this statement, however, the question of reliability of the theoretical calculations used for the determination of stresses under impact has not been taken into account. Consequently the allowable working stresses have to be diminished by introducing an additional "coefficient of uncertainty," the magnitude of which depends on the degree of knowledge we have of actual impact loading.

4—IMPACT BRITTLENESS

The author has not as yet discussed the question of impact testing of notched specimens in bending. Would it now be possible for the designer to use the coefficients of "impact ductility" obtained from such tests as additional factors in determining the allowable stresses? In order to obtain a clear answer to this, it is necessary to treat the question of the physical significance of such tests in a somewhat more detailed way.

It is known that there are materials which, while giving quite satisfactory results in static and even in impact tests for rupture, fail with scarcely any expenditure of energy during notched-bar rupture tests. This involves an element of danger, because among the various members of structures working under impact conditions, there are very many in which abrupt changes in form occur, and these can cause the same effect as notches (holes, grooves, keyway, blowholes, etc.). Now, what is the reason for such brittleness and why does it not manifest itself in other types of test?

Before answering this question let us consider the conditions that differentiate the impact test of notched samples from the static test, namely, the notch and the high speed.

As shown by Ludwik,\(^2\) during any plastic deformation the presence of a notch causes a special state of stress in the vicinity of the notch. In addition to the fundamental tensile stress caused by the external forces, there arise two other main tensile stresses representing the reaction of masses in the specimen not participating in the deformation but located in its vicinity. According to the hypothesis of maximum shear stress, the resistance to deformation is determined by the maximum difference of the principal stresses; therefore the presence of a special state of stress is accompanied to a certain extent by a rise of the stress diagram for the material at the base of the notch.

The speed, as was seen above, affects the stress-strain diagram of the tension in the same way—by further increasing the strength at a given deformation.

Referring to Fig. 5, let the curve $AB$ represent the actual tensile stress (i.e., the stress as referred to the actual area of the cross-section and not to the original area, as it is usually assumed). The instant of rupture is determined by intersection of this curve with another curve $CB$: the ordinates of which represent the actual strength of the material in various states of cold working.

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\(^1\) See S. Timoshenko, "Strength of Materials," 1932, part 1, art. 64, problem no. 9.

Unfortunately, we do not know the actual shape of this curve, and can only conjecture it, although recently Kuntze, using results of tests, has determined a number of points thereon.

On the basis of certain known facts and by theoretical reasoning it is safe to assume that the location of the curve $CB$, which depends only on molecular attraction, is neither affected by speed nor by the special state of stress (and even if this should be the case, it would have but little effect as regards the rise of the curve). Then, due to notch action and speed, the curve $AB$ will rise to the position $A'B'$, curve $CB$ remaining fixed, and continue rising until at a certain critical instant point $A$ of the curve reaches the point $C$. After this brittle rupture occurs, because the strength in shear will be greater than the strength in tension.

It is clear now how the simultaneous action of speed and notch will cause brittle rupture of a material which is entirely ductile when undergoing the usual tests. Soft steel (iron) behaves in this way as a result of improper heat treatment (coarse-grained, overheating), and chrome-nickel steel at a certain stage of annealing.

Variation in the conditions of the test (sharpness and depth of the notch, speed of the impact, temperature) will cause the same material to rupture sometimes as a brittle and sometimes as a ductile material. It is possible under given test conditions for two materials to prove equally ductile, while under more severe conditions one of them will experience brittle rupture and the other not. As a basis upon which to form an opinion regarding the tendency of a metal to brittle rupture, it is necessary to carry out a series of tests in which one of the factors affecting the rupture is varied.

The most convenient procedure is to use the temperature of the test as the variable parameter. While temperature has not as yet been mentioned as a factor, nevertheless it is responsible for the location of the curve $A'B'$ (Fig. 5). According to Schmid and Polanyi the dependence of the stress-strain diagram on temperature is an established fact and appears to be the fundamental reason for introducing the term "thermal plasticity." It is to be expected that with a lowering of the temperature the curve $A'B'$ will rise, and for materials which at room temperature are characterized by a ductile rupture, brittleness will appear at some lower temperature. Indeed, testing at varying temperatures gives a curve of the form qualitatively shown in Fig. 6 and having a more or less sharply defined "critical temperature range"—which is steep for a steel of low carbon content and flatter for carbon and special steels.

Fig. 7 shows the curves obtained in the author’s laboratory for two specimens of the same boiler plate (carbon = 0.10 per cent): curve I was obtained after a standard heat treatment and curve II after an artificial overheating (two hours at 1200°C). It is seen that the overheating and the resulting coarse-grained structure displace the "critical range" toward the range of higher temperatures, in this case of room temperatures. The position of the critical range appears to be the best criterion for the tendency of the material to brittleness.

If the idea advanced concerning the effect of the notch is correct, then brittle rupture must occur also under conditions when one of the factors is excluded—for instance, speed or notch—and the third one, temperature, is simultaneously increased. Indeed the author succeeded in obtaining, for the same boiler plate, brittle rupture of a cylindrical specimen (Fig. 8) on a pendulum testing machine. A similar exhibition of brittleness was observed by Sauerwald in monocrystal iron, which ruptured without deformation under impact at 98°C. Schwinning and Matthäus, on the other hand, used a notched specimen and replaced the impact test by a static one, obtaining also a brittle rupture, of course with a corresponding decrease in temperature (of the order of 30 deg). Speed alone, however, is sufficient only in the case of materials exceptionally inclined to brittleness—for instance, phosphorus iron containing more than 0.25 per cent phosphorus undergoes brittle rupture in impact tests of cylindrical samples at room temperature. Such materials, according to the old terminology, are said to be brittle in impact, although the

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38 W. Kuntze, Sonderheft 20, Deutsche Materialprüfungsanstalt, 1932.
41 F. Pettweis, Stahl u. Eisen, 1929, Heft 45.
42 By F. Witman.
44 F. Schoenmaker, First communication, NIATM, 1930A, p. 243.
5—Practical Significance of Impact Tests

From the foregoing considerations it is clear that the impact test of notched specimens, in spite of its necessity (at least for ferrous metals), does not afford results having any relation to the magnitude of the allowable stresses and should be used only for purposes of checking; it may either reject the material, or approve it for use under stresses allowable with static loading.

The establishment of conditions for the rejection or approval of material requires careful preliminary study of the material and working conditions of the structure in which it is to be used. Only a few general considerations have been presented here.

Obviously, the matter must be decided by the position of the critical range of cold rupture. Therefore the impact test cannot be satisfied by any single factor (type of sample, speed, and temperature of test), or even by two of them, as proposed by Moser; it must give the whole curve of the dependence, of the type indicated above, on the temperature. This of course introduces considerable complication into the impact test, and prohibits the employment of such tests in industry.

Perhaps in the future we shall learn how to determine in a reliable way the actual strength in rupture of a cold specimen and its yield point in impact at room temperature, raised by the influence of the notch; then, by introducing a definite ratio by which the first value exceeds the second one, we shall be enabled to ascertain the minimum tendency for brittle rupture. As long as this is not possible, as is generally admitted, there is no other way in which to proceed but that indicated above: namely, determine the whole curve or at least a few of its most characteristic points.

In order to set standard conditions for the test, it is necessary to consider the dependence of the temperature of cold rupture on the form of the notch, determining the gradient of the stresses and, consequently, the degree of progress of the special state of stress. Obviously the most severe conditions of the test result from using that notch which causes for the same material the highest temperature of the critical range; therefore it is desirable to base the selection of the standard sample upon a comparison of the critical temperatures given by various notches. According to a proposal by the author, material is to be prepared in this way for standardizing of the impact test at the NATI in Moscow.

FIG. 8 CURVES SHOWING DUCTILITY OF STANDARD AND OVERHEATED BOILER PLATE UNDER IMPACT: RUPTURE OF SPECIMENS WITHOUT NOTCHES

Under such circumstances, we can be sure that worse conditions do not occur in actual service of the structure.

In stating the concrete requirement as to the position of the critical range, there appear to be no difficulties when the interval is sharply defined and does not exceed 20 to 40 deg C, as in the case of steel of low carbon content. In such cases it is sufficient to state that the beginning of the drop in impact ductility shall not be lower than 20 to 40 deg C (depending on the temperature conditions of the structure); the exact form of the notch will then appear as an assurance of a certain amount of ductility.

More doubt will exist in those cases where the critical range is not distinctly defined, having an amplitude of the order of 200–300 deg (chrome-nickel steel), the drop in ductility beginning at a temperature considerably higher than that of the room (80–100 C). Here, obviously, it is not possible to exclude the structure from service in the range of initial brittleness. The question might easily arise: Is it not necessary to diminish the allowable stress in the ratio of the decrease of the impact ductility at the temperature of the structure in service as compared with the maximum value? Such a procedure, however, would not be correct. The experimental data available show that a decrease in the energy of deformation at a lowering of temperature occurs on account of the decrease in extension, and not on account of the strength of material, which remains approximately constant; therefore, in assuming equal allowable stresses there is no decrease in the factor of safety. Only the degree of ductility drops, and if this drop is not large (say, 10 per cent), then it is a matter of judgment of the designer whether such a drop is admissible or not.

A necessary condition for its admissibility is the evidence given by the diagram of the test, that the drop in ductility occurs in a sufficiently slow manner and that complete brittleness occurs only at the lowest temperatures.

6—Conclusions

Summarizing the considerations already stated, we arrive at the following conclusions:

1. The yield point and tensile strength of a material under impact are always higher than those at low-speed tests, the yield point rising more rapidly than the strength.

2. The working conditions of the material under impact are less favorable than those under static action of a force.

3. As far as these two circumstances compensate each other, it is recommended that in calculations for impact the same allowable working stresses be used as for static conditions, employing the data obtained in static tests. (Neglecting, however, consideration of the reliability of the calculations.)

4. The impact test for notched specimens can be used only as a check test, approving the use of a material or rejecting it; in determining the allowable stresses to employ, however, the results of such tests are of no value.

Discussion

E. Dillon Smith. There are certain data in the paper that seem to offer promise of revealing new truths about impact. The author's symposium is the expression of thought of the impact workers of European origin. The writer will try to bring out certain thoughts not mentioned by the author, as well as to present a critical review of some of the data and expressions in the paper.

Should not the conception of impact be based on the fundamental relation of force? This relation is:

\[ F \cdot V^2 = \text{const} \]

W. Schwinning and K. Matthaes, footnote 26; see also that the diagrams of the static tests are the most reliable ones.

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In this, it would seem evident that the force is dependent upon the deceleration of the hammer (tup or bullet). An example of this idea may be given in the following manner. A common hammer is allowed to strike a steel beam. The hammer hits the beam with a velocity of 200 ft per sec, the hammer weighing 1.93 lb. Further, let us assume that the hammer will come to rest within 0.001 sec, which is not at all improbable, and in fact might come to rest in a much shorter time. The deceleration of the hammer may be a maximum at 200,000 ft per sec per sec. Substituting, in [1], we have \( F = \frac{(1.93/32.2)}{200,000} = 12,000 \) lb.

If the assumption of a shorter time had been taken for the hammer to decelerate, it would be evident that this impact force is considerably greater than that just given. Further, place the force value computed by [1] upon this beam as a static load of 12,000 lb. Certainly we can conclude that the effects of the impact and the static forces are not the same.

The rapidity of the application of the force imparted by the hammer and the inertia of the beam will produce effects that will be varied. The result will be further qualified depending on the load being applied at a rate greater or less than the natural period of the specimen under observation.

Taking an average value from Table 1, we learn that a beam withstanding an impact loading (how determined, the author fails to mention) of 15,583 lb will not fail unless a static load of 10,000 lb is exceeded. Would not a moment's thought indicate that this condition is not valid for all cases, depending on the speed of the application of this 15,583 lb, the inertia, and the period of the beam?

Tables 1 and 2 also have been examined as to the agreement of the results secured by the different investigators. Although "the numerical results of Table 1 are valid," that is no indication that such numerical results are a correct measure of the specific phenomenon being studied. The results shown in this table are found to be homogeneous, or within highly significant agreement, although the author seems to question them. The commensurate data of Table 2 are also homogeneous; however, there is reason to believe that more data are needed to indicate that they are as significant as those data listed in Table 1.

Figs. 7 and 8 are to be questioned from the standpoint of general theory. Especially in Fig. 8, it seems illogical to connect the upper and lower sections of the dots with vertical lines, as is shown. One reason for this disagreement is based upon the writer's Fig. 9 for quartz piezoelectric crystals. (Although, for this case, tuned oscillatory circuits are one explanation; but, notwithstanding, there is a change due to temperature shift.) The ordinate has been translated to a function of impact loading. Another reason for the writer's disagreement to the treatment of data in this manner is that most generally reaction and growth curves follow some definite law. Fig. 7 follows a natural change by a law controlled by a mathematical logistic curve. But, in the case of Fig. 8, it would seem that there are two different and distinct changes in the state of the boiler plate—that there are two growth curves of the logistic type.

Referring to the two methods of determining impact forces, there are certain facts that should be mentioned about the schemes. Under class 1 the author mentions that considerable error is present when graphic means of double differentiation are resorted to for obtaining the acceleration curve from the space-time curve. The United States Bureau of Public Roads seems to have concluded differently.13 But if this is not satisfactory enough, why not revert to a translation of the space-time curve into a mathematical equation and then take the second derivative? At the same time it is possible to compute the error in any series or equation by which we wish to express the observed condition. The success of this method rests with one's ability to select the relation that most nearly fits the space-time curve plotted by the hammer.

As an example of this mathematical method, assume that the space-time curve is a straight line (which it is not—always being of second degree or higher, but its trend at certain sections can be shown linearly), and let us investigate the error.

### Figure 9

**Effect of Temperature on Damped Oscillations in a Quartz Piezoelectric Crystal**

The standard error of the straight line

\[ f(x) = y = a + bx \]  

becomes

\[ \sigma_y = \epsilon \left\{ \sigma_a^2 \left( \frac{\partial f}{\partial a} \right)^2 + \sigma_b^2 \left( \frac{\partial f}{\partial b} \right)^2 + 2\sigma_a \sigma_b \left( \frac{\partial f}{\partial a} \frac{\partial f}{\partial b} \right) \right\}^{\frac{1}{2}} \]  

The partial derivatives are

\[ \frac{\partial f}{\partial a} = 1, \quad \frac{\partial f}{\partial b} = x \]

Now since we can take the origin at the middle of the range of the curve of \( x \), \( \{a, b\} = 0 \). Substituting, we obtain the hyperbola

\[ \sigma_y = \epsilon \left\{ \sigma_a^2 + \sigma_b^2 x^2 \right\}^{\frac{1}{2}} \]  

Where

\[ \sigma_a^2 = \frac{\Sigma x^2}{n\Sigma x^2} = \frac{1}{n} \]  

\[ \sigma_b^2 = \frac{n}{n\Sigma x^2} = \frac{1}{n\Sigma x^2} \]
and where

\[ \epsilon = \left( \frac{2(y - y')^2}{n - 1} \right) \]

for our specific case; \( y \) being the observed reading and \( y' \) the computed or mean value at that same value of \( x \); \( n \) is the total number of readings observed.

This can be shown graphically, as given in Fig. 10. Also, the results of a parabola, treated similarly, are shown in this same Fig. 10. In the case of the example cited, it is noticed that the error is the least at the point most concerned in our investigation, the point of maximum acceleration.

The difference or the tangential method of obtaining the results graphically seems to be as reliable as any mathematical method, and often better when only a few cases are to be studied; as the evaluation of the constants in the mathematical equations is rather a long job. Since a certain fact has been mentioned about class 1, let us turn attention to class 2.

In the case of the self-recording dynamometer, the piezoelectric cell and the cathode-ray oscillograph device have been described. The writer does heartily agree that the author has a most excellent basic set-up for the measurement of impact directly, but he has not carried this idea far enough, nor does it seem that this device will be able to give data on impact when impact forces delivered by the hammer are above those forces at which the piezoelectric crystal will rupture. The rupture of the crystal will occur before the rupture of an iron or steel specimen. It is interesting to note that reference 34 in Z.V.D.I. uses this same idea for tool loads, but the loads are not severe enough to rupture the crystal. Of course, below this point of rupture of the crystal possibly excellent data should be secured.

For quartz crystals we find that

\[ de = kV \]

where the differential of elongation varies as the change in the developed voltage upon its proper faces, being a maximum for the \( x \) or perpendicular cut. That being the case, the voltage developed by the external forces upon the crystal give a direct measure of the force applied. This has been shown in Fig. 4 as the ordinate. Incidentally, these photos are not those given by a
cathode-ray oscillograph; unless the beam is deflected in only one plane and recorded on a passing film, which is not customary. It is noticed that these photos are not analyzed in this paper. Before they may be analyzed, calibration of the device is necessary. How does the author propose to calibrate his device? May the writer suggest that in reality the author has only one fundamental method—that of taking an autographic space-time curve of the hammer at the same time the oscillograph records the developed voltage from the crystal. But this is practicable only for impact forces less than the rupture point of the crystal.

It would seem that "an additional 'coefficient of uncertainty' " would not be required if the impact research is carried on under a proper scientific method and employing its best tools.

Perhaps it would be interesting to inspect a curve illustrating these things. Such a space-time and derived curves are shown in Fig. 11. A 34-lb hammer has fallen 3 in. on a 1/8-in. by 1/8-in. by 30-in. steel beam to record these data. It will be seen that the striking velocity of the hammer was 4.35 ft per sec with a maximum deceleration of 512 ft per sec per sec, with a deflection of a maximum of 0.270 in. Computation gives an impact force of 540 lb hitting the beam. In this particular case, a numerically equal static force would produce a deflection of approximately 20 per cent less.

R. V. Southwell. Professor Timoshenko gave an account, with illustrative diagrams, of work conducted recently at the University of Oxford, England, by Mr. J. H. Lavery in collaboration with Prof. R. V. Southwell. This work was based upon two underlying ideas: First, that the stress distribution imposed by tests of the Izod and Charpy types is too complex to be estimated by theory, and involves an undesirably large amount of plastic distortion in regions not immediately adjacent to the surface of fracture; second, that machines which employ a rigid pendulum can transmit waves of stress away from the machine and to "earth," and energy so transmitted is included (incorrectly) in the estimated "work of fracture."

The first of these contentions is illustrated by Figs. 12a and b, and Fig. 13 shows the specimen and method of loading which has been adopted at Oxford. A hardened yoke transmits the blow of the hammer to the specimen, which accordingly is subjected to "four-point loading,"—i.e., to a uniform bending action (unaccompanied by shear) in the region adjoining the notched section. On account of the large bearing areas, there is little or no penetration, and no appreciable plastic distortion is involved, except at the actual surface of separation. The specimen can be formed entirely in the lathe, since the "notch" is a concentric groove, and accordingly it is cheap to manufacture.

The objection made to machines of the rigid-pendulum type is obviated in the Oxford machine by suspending both "hammer" and "anvil" on flexible cords so that they can move without rotation. The blow occurs at the center of gravity of anvil, hammer, and specimen, and in a direction perpendicular to the cords; thus no energy can escape from the machine, and the energy involved in stress waves (which will be of the longitudinal type) is very small because the stresses are low. Figs. 14 and 15 illustrate the construction of the machine, and Fig. 16 is a photograph of the hammer, anvil, and specimen.

Comparative tests were made, using exactly the same shape of
specimen and method of loading, both in the Oxford machine and in a standard Izod machine suitably modified and provided with devices to permit specially exact measurement. Fig. 17 shows the results obtained with a nickel-chrome-molybdenum steel, in 50 tests made from material of a single batch, specimens being tested in the two machines alternately. The uniformity of the results is thought to be satisfactory, and a consistently lower figure for the work of fracture is obtained in the Oxford machine, as would be expected if the contention is correct that energy transmitted to earth has been wrongly included in the Izod figures.

Fig. 18 exhibits the results of 59 tests made on specimens having different areas under the notch, but otherwise identical. The work of fracture is closely proportional to the area of fracture. This result is explained by the concentration of plastic distortion within the material which immediately adjoins the fracture. It is thought that the slight rise of the curve (with increasing area) can be explained, and the results are evidently indicative of a satisfactory dimensional law.

These 59 tests relate to specimens made with as sharp a notch...
as possible. On the same diagram are shown (by two crosses) results obtained from specimens made with a notch of 0.01 in. "root radius," but otherwise similar. It appears that the work of fracture is not sensitive to small variations in the "root radius," so that ordinary errors of workmanship will not have important consequences. This result is of course important from a practical standpoint.

MASON A. STONE. In view of the difficulties in determining the stresses occurring under impact which the author discusses in section 1 of the paper, his conclusion in section 6, paragraph 1, which summarizes the figures in Table 1, are very comforting to designers who have to provide for impact. The paper is valuable not only for its content but for the bibliography of references to its subject contained in the footnotes—which the writer would have welcomed some five years ago when called upon to design a two-story outdoor switching station with the oil circuit breakers carried on the upper floor.

At this time the only discussion of a beam to carry a transverse impact load which he was able to find was in Merriman's "Mechanics of Materials" (page 335 of the 1906 edition). As the footnotes do not give many references to the subject matter of section 1, either to the dynamic or empirical method, and as the latter may turn out to be a very poor guess indeed, perhaps an outline of methods followed and results obtained in the design of a typical floor beam and girders of this switching station may be in order. It will serve to illustrate the practical advantages in the assumption of a high working stress in the design of beams under transverse loading and the dangers of trying to provide for impact by the addition of an arbitrary percentage to the static load.

The manufacturers of the circuit breakers advised us to provide for an impact load from the operation of the breakers under a short circuit equal to that which would result from the fall of twice the weight of the breaker (exclusive of the frame) through 2 in. This weight was 17,900 lb. Obviously, a blow of 6000 ft-lb would have welcomed some five years ago when called upon to design a two-story outdoor switching station with the oil circuit breakers carried on the upper floor.

These values are only valid for a simple beam struck at the middle, as they depend upon the form of the deflection curve and the velocity of the beam at the load after impact. The formulas are developed upon the assumption that the shape of the elastic curve is the same for impact as for static loading.

The breaker was carried upon a frame standing upon four legs, which were carried upon two floor beams 20 ft long. The length of the breaker was 13 ft 4 in. Besides the loading from the breaker, the floor beams were designed for the load from a floor panel 6 ft 4 in. wide of 4 1/2-in. concrete and a snow load of 25 lb per sq ft.

The value of \( \eta \) for a beam carrying two loads equidistant from the ends one-sixth the length of the beam was determined to be 0.69.

The impact stresses in this case amounted to four times those from the quiescent dead and live loads.

Inasmuch as these stresses occurred only for an instant, it was considered justifiable to adopt the practice generally followed in the case of stresses from wind load of adding 50 per cent to the allowable fiber stress of 16,000 lb per sq in.

The assumption of a high permissible fiber stress is desirable, because the usual expedient to deepening the beam to reduce stresses does not work out very well in the case of impact. A stiffer beam may result in higher fiber stresses. This was discovered many years ago when the experiment of deepening the rail section was followed by an increase in the number of broken rails in service. In designing beams for impact, therefore, flexibility rather than stiffness is desirable. This may be obtained by the use of cover plates so applied that the fiber stresses are approximately constant throughout the length of the beam, as a beam of uniform cross-section may have twice the stiffness of a beam of cross-section varied so that the fiber stress is constant throughout.

This may be illustrated by the design of the girders into which the floor beams framed. A 24-in. 110-lb Carnegie I-beam 23 ft 6 in. long was first tried, which gave a stress from impact alone of 25,200 lb per sq in. The girder finally decided upon was composed of a 18-in. 86-lb Carnegie I-beam and two 12-in. by 3/8-in. cover plates 11 ft 6 in. long and two 12-in. by 3/8-in. cover plates 5 ft 9 in. long.

This resulted in a dead and live load stress of 5360 lb per sq in. and 15,500 lb per sq in. from impact, or 20,860 lb per sq in. for the combined stresses. Of course the determination of \( \eta \) for such a built-up section with a variation in the moment of inertia is somewhat troublesome.

The explanation of the action of the boiler plates as shown in Figs. 7 and 8 was interesting. This phenomenon is well understood by boiler makers, as a plate which is being flanged goes through a temperature range, which is known to the smiths as a "blue heat" from the color of the oxide formed on the surface of the metal, at which the plate cannot be worked because of brittleness, although ductile above and below this range.

This may be of interest to the designers of power-plant piping if superheat temperatures continue to increase.

The following fiber stresses were obtained: From the dead load of the breaker, 2130 lb per sq in.; from the dead load of beam, slab, and snow, 2600 lb per sq in.; from impact, 19,000 lb per sq in.; total combined stresses, 23,700 lb per sq in.

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Stability of Thin-Walled Tubes Under Torsion

BY L. H. DONNELL, PASADENA, CALIF.

A THEORETICAL solution is developed for the torsion on a round thin-walled tube at which the walls become unstable and buckling occurs.

\[ S = \text{critical shear stress, assumed uniformly distributed} \]
\[ E, \mu = \text{Young's modulus and Poisson's ratio (0.3 for engineering metals)} \]
\[ l, t, d = \text{length, wall thickness, and diameter of the tube} \]
\[ A = (1 - \mu^2) \frac{Sli}{Et} \quad H = \sqrt{1 - \mu^2} \quad J = \frac{1}{\sqrt{1 - \mu^2}} \]

The critical stress, for short and moderately long tubes, is given by the formulas:

\[ A = 4.6 + \sqrt{7.8 + 1.67H^{1/4}} \quad (\text{clamped edges}) \]
\[ A = 2.8 + \sqrt{2.6 + 1.40H^{1/4}} \quad (\text{hinged edges}) \]

It is assumed that end cross-sections of the tube remain circular and plane, that "clamped" edges are held perpendicular to these cross-sections, while "hinged" edges are free to change their angle with the cross-sections. It is found to be immaterial whether or not the ends of the tube are free to move as a whole. The buckling deformation is found to consist of a number of circumferential waves which spiral around the tube from one end to the other. For tubes of ordinary length-diameter ratios, with clamped edges, the number of waves \( n \) and their spiral angle near the middle of the tube \( \theta \) are approximately:

\[ n \approx 2.6J^{-1/4}, \quad \theta \approx 1.5 \frac{d}{l} J^{1/4} \text{ rad.} \]

If \( J > 8 \), that is for long, slim tubes, \( n \) is always 2, and the critical stress is nearly independent of the edge conditions and is given by the formula: \( A = 0.77 \sqrt{(l/d)H} \), which is nearly the same as a formula found by E. Schwerin in 1924.

To check these theoretical results the author has made more than fifty experiments; in addition, the results of fifty or sixty more experiments have been published by the National Advisory Committee for Aeronautics and others. All available tests give values for the failure stress somewhat lower than the values for critical stress predicted by the foregoing formulas. The experimental values average about 0.75 of the theoretical, with a minimum about 0.60 of the theoretical. These relations hold consistently over an enormous range of sizes, proportions, and materials. The form of the buckling deflection, as measured by the number and angle of the waves, is found to check closely with that predicted by the theory.

It is therefore reasonable to suppose that the discrepancy between the theoretical and experimental stress is due chiefly to initial eccentricities and other defects unavoidable in an actual tube, as well as to the fact that all tests were performed with "clamped" edges, while it is well known that a perfect clamped edge is impossible to attain. By multiplying the foregoing expressions for \( A \) by the factor 0.75 or 0.60, we can obtain expressions for the average and minimum resistance to buckling to be expected from an actual tube.

In developing the theory, the assumptions are made that the material obeys Hooke's law, that the tube is exactly cylindrical, that the thickness is small compared to the radius (as it must be if buckling is to occur before failure of any available material), and that the deflections are small compared to the thickness. With these assumptions, the differential equations of equilibrium are derived in the following form:

\[ \frac{E}{12(1 - \mu^2)} \frac{d^4v}{dx^4} + 4 \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x^2 \partial y} + 2 \frac{S}{E} \frac{\partial^3 w}{\partial x \partial y \partial z} = 0 \]
\[ \frac{d}{2} \frac{\partial^2 v}{\partial x \partial z} = - \left( 2 + \mu \right) \frac{\partial^2 v}{\partial x^2 \partial z} - \frac{\partial^3 w}{\partial x \partial y \partial z} \]
\[ \frac{d}{2} \frac{\partial^2 u}{\partial x \partial s} = - \mu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^3 w}{\partial x \partial y \partial z} \]

where \( v \) = \( x \) and \( s \) are axial and circumferential coordinates, and \( u, v, \) and \( w \) are axial, circumferential, and radial components of displacement, respectively. These equations are much simpler than those usually given, chiefly because many of the items commonly taken into consideration are of negligible importance for this and many other cases, as is shown in the complete paper.

The solution obtained is an exact solution of these equations of equilibrium and of complete boundary conditions, for the two extreme cases when the length-diameter ratio is zero and infinite and is a good approximation for intermediate cases.